CIRCLE MAPS AS A SIMPLE OSCILLATORS FOR COMPLEX BEHAVIOR: II. EXPERIMENTS

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ABSTRACT

The circle map is a general non-linear iterated function that maps the circle onto itself. In its standard form it can be interpreted as a simple sinusoidal oscillator which is perturbed by a non-linear term. By varying the strength of the non-linear contribution a rich array of non-linear responses can be achieved, including wave-shaping, pitch-bending, period-doubling and highly irregular patterns. We describe a number of such examples and discuss their subjective auditory perception.

1. INTRODUCTION

Circle maps are a particularly simple yet rather general example of a mapping that exhibits many important aspects of complex dynamical behavior. A circle map is capable of demonstrating such behaviors as mode and phase-locking, period doubling and subharmonics, quasi-periodicity as well as routes to chaos via repeated period doubling or via disruption to quasi-periodicity [1].

Circle maps are also attractive because they have served as an important “simplest case” example of iterated dynamics in the study of these dynamics among mathematicians and physicists. They also are related to already proposed sound synthesis methods that worry about introducing functional iterations or non-linearities.

The circle map is particularly suitable for the study and generation of sustained undamped sounds as the map confines the space of possible iterations exactly to functions of this nature by construction.

This paper is the second in a series of papers describing the circle map for sound synthesis purposes. The aim is to describe the properties in a systematic fashion to allow easy use. Additionally it is an example of classifying synthesis methods with respect to perturbation of parameters. The purpose of this paper is to discuss computational results of the circle map. Specifically we are interested in properties which are relevant for the use of the circle map for sound synthesis. The basic properties of the circle map were already described in [2] and we will only give a brief review of the method here and refer the reader to this reference for more detailed introductory discussions.

2. BACKGROUND

Non-linearities have played an ongoing important role since very early. Risset introduced [3], and Arfib and Le Brun and refined wave-shaping, a method where a pre-existing signal would be fed through a non-linear function, hence modifying the sound [4, 5]. The method is able to create complex though generally only perfectly non-chaotic, periodic signals and the control is well understood.

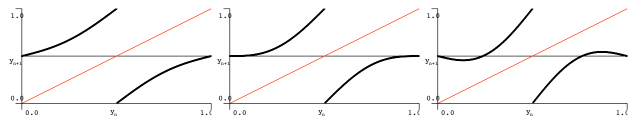

Figure 1: *The mapping $y_{n+1} = \phi(y_n)$ for the standard circle map at values $k = 0.5$ (left), $k = 1.0$ (center) and $k = 1.5$. The horizontal line at y_{n+1} marks the boundary between invertible and non-invertible maps. At $k = 1$ the map becomes tangent to this limit line. For $k = 1.5$ the line is crossed more than once and the map is non-invertible.*

Chaos itself became a focus of attention in the late 80s and early 90s. The use of iterated functions that can lead to rich nonlinear and chaotic behavior falls into to broad categories: (1) The use of periodic pattern in the generation of music structure and (2) for direct sound synthesis purposes. Within the first category Pressing studied logistic maps [6]. Gogins [7] investigated randomly switched sets of functions in his iterations. Bidlack introduced physically motivated maps of either dissipative or conservative character using Lorenz-type and Henon-Heiles type iterations [8]. The second category was developed by Truax [9] and Di Scipio [10, 11] motivated directly by iterated maps. DiScipio considers what he calls the sine map, an iterated sinusoid without coupling to a linear function. Rodet considered Chua’s network and its time-delayed extension for sound synthesis who also draws connections to nonlinearities in a physical context [12, and references therein]. Dobson and Fitch considered iterated complex quadratic maps [13] experimentally. Manzolli et al consider a set of two-variable iterations which are variations of the so-called *standard map* which in turn is related to the circle map [14]. Recently Valsamakis and Miranda consider a family of two variable coupled oscillator with sine waves in the feedback loop [15] The most widely cited reference of chaos theory is the computer music literature is [16]. It does contain a description of the circle map but gives little interpretation or motivation of the map. Maybe for the lack of emphasis of the specific properties of the circle map, it has not been widely considered as a desirable model for iterative synthesis and sequence construction in the above mentioned literature.

3. ITERATED MAPS FROM THE CIRCLE TO ITSELF

The most general form of the circle map is

$$y_{n+1} = \phi(y_n) \tag{1}$$

where the defining property is that ϕ is a mapping from the a bound interval to a bound interval of the same size, or alternatively

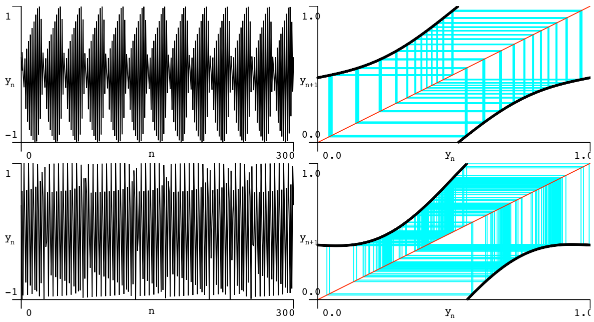

Figure 2: *Top: Beating pattern at* $\Omega = 0.475, k = 0.6$*, Bottom: Jitter pattern at* $\Omega = 0.4, k = 1.1$*. Left shows the waveforms, right shows the mapping function and the iteration trace. The diagonal line are the fixed point locus.*

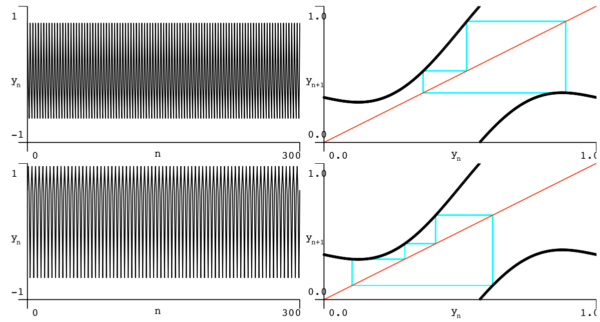

Figure 3: *Top: First one two bistable simple orbits at* $\Omega = 0.33, k = 1.42$*, Bottom: Second of two bistable simple orbits at* $\Omega = 0.33, k = 1.42$*. Left shows the waveforms, right shows the mapping function and the iteration trace. The diagonal line are the fixed point locus.*

of a periodically repeating interval. A periodically repeating interval is topologically equivalent to a circle, hence the name of the map.

If we want to model a perfectly sinusoidal oscillator that is perturbed by some coupled non-linear function, this turns into:

$$y_{n+1} = \left(y_n + \Omega - \frac{k}{2\pi} f(y_n)\right) \mod 1 \tag{2}$$

where Ω is a constant that is the fixed angular progression of the sinusoidal oscillator, and k is the coupling strength of the non-linear perturbation $f(\cdot)$. y_0 is the starting phase. In principle, the choice of $f(\cdot)$ is very flexible and examples of discontinuous functions can be found in the literature as well as smooth cases. We will consider a number of examples later. The canonical theoretical example is the *standard circle map*:

$$y_{n+1} = \left(y_n + \Omega - \frac{k}{2\pi}\sin(2\pi y_n)\right) \mod 1 \tag{3}$$

In order to study the long-term behavior of the iterated map $\phi(\cdot)$ we can look at the *winding number*

$$W = \lim_{n\to\infty} \frac{y_n - y_0}{n} \tag{4}$$

which measures the average angle added in the long term. If this added angle notated over the interval $[0, 1)$ is a rational number p/q with $p, q \in \mathbb{N}$ then after q iterations we will have a recurrence and hence the map is periodic. Irrational winding numbers are called *quasi-periodic*.

Throughout this paper we will call a response *singular* if any perturbation to the parameters results in a qualitative change of the response, otherwise it is *stable*. We will call a response *generic* if most, but not necessarily all responses under variation of one or more parameter stays qualitatively the same.

We will call a closed path an *orbit*. For orbits where the number of iterations until repetition is known we say the path is an n*-orbit* where n is a positive integer. We call a path *regular* if it is periodic or quasi-periodic. Highly irregular patterns will loosely be called *chaotic*.

Ω is a phase progression, but of course is essentially the frequency of the unperturbed oscillator which is calculated as $f_\Omega = \Omega \cdot S$ where S is the sampling rate, or time interval between two time steps for $\Omega \in [0, 0.5]$. Therefore we will call Ω a frequency throughout this paper. If $\Omega > 0.5$ we get aliasing and the effective frequency decreases again, which opposite phase sign.

4. OBSERVED WAVEFORMS IN THE PARAMETER PLANE (Ω, K)

The parameter k defines the strength of the influence of the non-linear term is on the overall iteration. If k is small or vanishes, we get behavior close to or equivalent to a pure sinusoidal oscillator. With increasing values of k the non-linear term starts to dominate. The specific change in behavior of k also depends on the choice of the linear oscillator frequency Ω as we shall see below. In this section all rendered examples are for the standard circle map of equation 3.

Waveforms were generated by the result of a given iteration n into a sine function:

$$Y_n = \sin(2\pi y_n) \tag{5}$$

4.1. Quantitative Change with Varied Coupling k

In the case of the standard circle map the behavior with respect to k can be be roughly classified into three regions. If $|k| < 1$ then the transfer function is invertible, that is a unique y_n maps to a unique y_{n+1}. At $|k| = 1$ a self-tangency forms and at $|k| > 1$ the mapping isn't invertible anymore as multiple values of y_n map to the same y_{n+1}. This can be seen in Figure 1 where the left case of $k = 0.5$ no self-overlap parallel to the y_n-axis, where as the right case clearly does. We will call the case $|k| < 1$ example of *small* k whereas the case of $|k| > 1$ constitutes examples of *large* k. The case $|k| = 1$ is singular and not of practical interest.

For the same reason, for values of $|k| < 1$ the sign simply means an inversion of the waveform about the line $y_n = 0$ while for $|k| > 1$ one gets qualitatively different behavior for positive and negative coupling constants.

4.1.1. Behavior for Small k

For small k one can already see non-linear behavior. Most prominent are waveform deformations, reminiscent of wave-shaping or

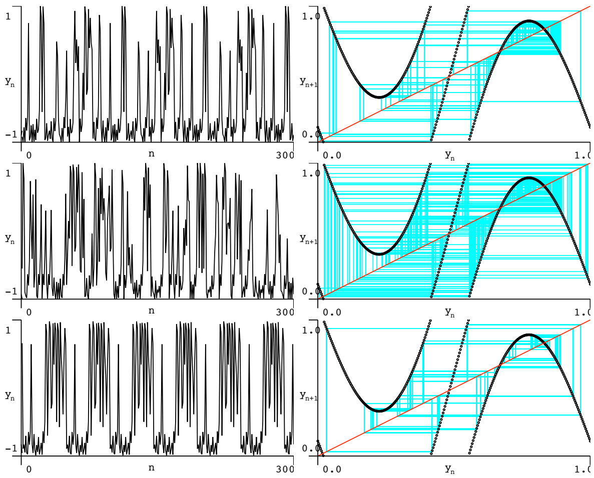

Figure 4: *Sensitivity to iteration start position y_o at $\Omega = 0.11, k = 6.4$. Two orbits (top, bottom) are stable, one (center) is chaotic. All orbits are non-singular.*

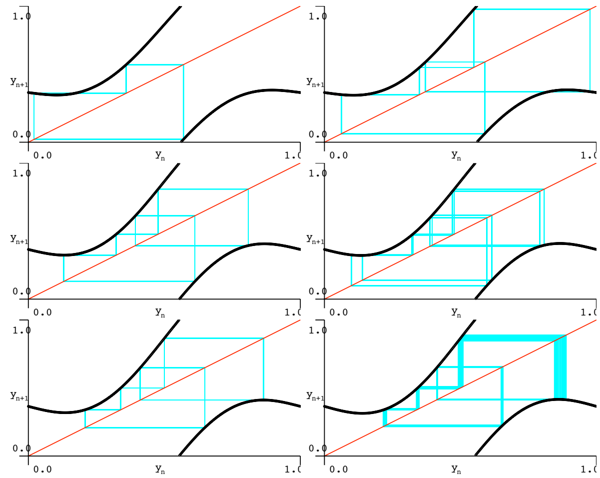

Figure 5: *Left to right, top to bottom: Mapping function and iteration trace of period doubling and first complex pattern formation for $\Omega = 0.365, k = 1.26, 1.28, 1.48, 1.5, 1.52, 1.56$. The diagonal line are the fixed point locus.*

phase distortion synthesis [17]. Also beating can occur. An example of both these phenomena can be seen in the top example of Figure 2.

Another phenomenon of practical interest is the relation of the mapping function to the line of fixed points. In all figures, the center diagonal line depicts the fixed points of the map, that is points where

$$\phi(y_n) = y_{n+1} = y_n = y_f. \quad (6)$$

Clearly once a mapping hits a fixed point it will stick to this point. With respect to fixed points a number of basic properties can again be observed: If the relative angle of the intersection at the fixed point y_f is mild , that is if $f'(y_f) < 1$ then starting points of the iteration in the neighborhood of that fixed point will converge to the fixed point y_f [18, p. 482 for a related discussion for the logistic map]. This has practical implication for the qualitative behavior of the standard circle map. As k is increased, the amplitude of the sinusoidal non-linearity is increased and hence the slope of the intersection that is possible. For small k the slope can generally be expected to be small and hence, generically, $f'(y_f) < 1$ when there is an intersection with the fixed point line. Convergence to fixed points translate into rapid decay to silence and hence these cases are not of practical interest. These silent regions are likely for small values of k and occur whenever the iterated function $\phi(x)$ intersects the fixed point line. If an intersection happens is a function of both Ω and of k. Ω defines a constant offset from the fixed point line, hence larger values of Ω translate into larger possible values of k before intersections happen.

For intuition one may say, that oscillatory waveforms for small k are limited by the frequency Ω. For low frequency oscillation one has less room for adding non-linear contributions than for higher frequencies.

Qualitatively the same is true if other functions than sinusoids are used as the iteration nonlinearity $f(\cdot)$. The defining property is the angle of the function at an intersection with the fixed point line.

4.1.2. Behavior for Large k

As said earlier, for large k, the map is non-invertible. Also from the previous discussion, the map function $\phi(\cdot)$ is much more likely to intersect the fixed point line. Every actual intersection point the map with the fixed point line:

$$\phi(x_n) = x_{n+1} = x_n \quad (7)$$

forms a sparse set of point for which the iteration results in a constant wave form. Due to the sparsity this does not have many practical implications, though it is important to know that by accident a poorly chosen starting point can result in silence.

For small k silence is a rather frequent response. However, with increasing k, the slope of, for example, the standard circle map at intersection points successively increases. Eventually it exceeds the slope necessary for convergence and non-attracting orbits away from the intersection fixed point become possible again. In fact silence becomes rare with very large k.

This is especially interesting for low frequencies Ω, where the potentially interesting responses of the standard circle map are rather confined. This confinement is lifted for sufficiently large k and the non-linear response also observed for larger Ω can be found equally for low frequencies.

4.2. Some Observed Phenomena

This section discussion some observed phenomena using the circle map, with an eye towards phenomena that are relevant for synthesis. For large k the behavior can change drastically for small perturbations of the parameters and hence at least naive control of the map is very difficult. The ultimate goal is to define ways match parameters with specific phenomenological responses. This task is however, future work and here this paper confines itself to giving examples of parameters that illustrate certain properties are seem interesting for sound synthesis.

Examples of phenomena can be found:

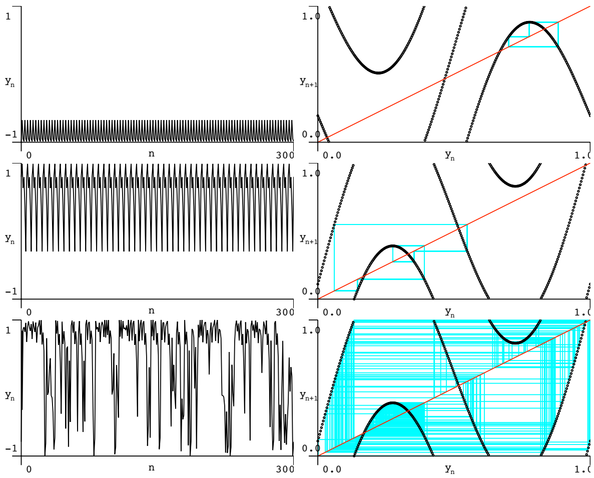

Figure 6: *Simple orbits for large k. Top: A stable simple orbit at $\Omega = 0.195, k = 5.8$. Center: A singular orbit at $\Omega = 0.11, k = -6.4$. Bottom: Chaotic regime for all other initial values of y_n at $\Omega = 0.11, k = -6.4$.*

- Mode locking: Mode locking is a feature for small k and becomes more prominent with increasing k [2, for some more details]. Mode locking means that around frequencies with rational frequencies, the oscillator tends to lock to those frequencies.
- Wave distortion: Wave fronts can be distorted from a sinusoidal form. Qualitatively this is similar to wave-shaping or phase distortion synthesis. This can be seen in the top example of Figure 2.
- Beating: A wave form is overlaid with a lower frequency envelope. The top part of Figure 2 shows a beating pattern for fairly mild coupling.
- Jitter: Wave fronts have disrupted phases or jittery phases. The bottom part of Figure 2 shows a jitter pattern.
- Pseudo-Noise: Wave fronts become strongly disrupted becoming perceptually noisy. All of Figure 4 shows this, as well as the bottom example of Figure 6.

Additionally, for large k the map becomes possibly but not necessarily sensitive to initial values of the iteration. Figure 3 shows such a case. A single point at $\Omega = 0.33$ and $k = 1.42$ in the parameter plane, which is bistable. This means that for different initial points of the iterations y_0 two different stable orbits are possible. Those are however stable. The first has a 3-orbit and the second is a 4-orbit. Such a bistable configuration is rare. A more typical example of the sensitivity to initial values of the map is shown in Figure 4. All three states in this case are for the same point in the parameter plane but with different starting positions y_n all three orbits are stable, hence can easily be achieved with a wide variety of starting points. But only two of the points are regular.

An important path to chaotic behavior is the successive bifurcation of resonant frequencies with increasing non-linearity k. The map goes through successive steps where the period doubles, which indicated by a doubling of the length of an orbit. An example of such a bifurcation is illustrated in Figure 5. We see that a single orbit separates into two similar but connected orbit, hence doubling the period.

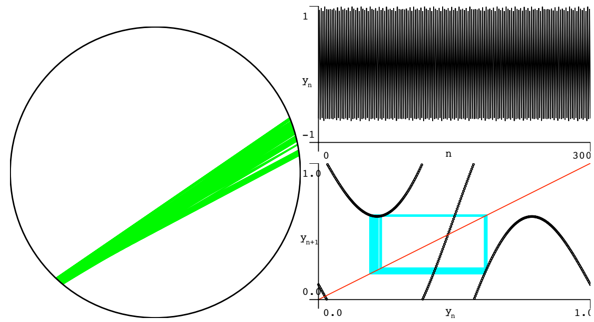

Figure 7: *Refocusing: The map only occupies a restricted angular segment of the circle and repeatedly refocuses on this area for $\Omega = 0.11, k = 4.6$. The fine structure of the area is sensitive to initial conditions. Right: Values of the iteration y_n are mapped onto a circle to show the confinement of the map more clearly. Top left: Wave form of the response. Bottom left: The mapping function and the iteration trace.*

Simple patterns can be observed even for large k. Figure 6 shows such simple orbits. One is stable for all initial values. The other is singular, embedded in an otherwise chaotic regime.

Figure 7 shows refocusing behavior. A beam of iterations only occupy a narrow area of two segments on the circle and repeatedly refocus on this area. Interestingly, the fine structure of this beam is not stable with initial values of the iteration.

5. VARIATION IN THE NON-LINEAR FUNCTION

The function $f(\cdot)$ of equation (2) significant influence on the actual wave form. This is often not emphasized in the mathematical literature, because one can show that despite variation of this transfer function only certain specific properties of the function have an influence how period-doubling and eventually chaos occurs. Specifically the degree of the turning points of the map is such a determining factor, but the shape of the function away from turning points is not [1].

However this change in function still has drastic influence to the sound and also change the qualitative behavior if turning points change. For this reason we describe a few experiments with variation of these transfer function.

We consider three functions in addition to the sine for $f(\cdot)$ in equation (2):

$$f(y_n) = \begin{cases} 4 \cdot y_n & \text{if } 0 \leq y_n < 1/4 \\ (1/4 - y_n) \cdot 4 + 1 & \text{if } 1/4 \leq y_n < 3/4 \\ (y_n - 3/4) \cdot 4 - 1 & \text{otherwise.} \end{cases} \quad (8)$$

$$f(y_n) = \begin{cases} \frac{y_n + T}{1 + 2 \cdot \epsilon \cdot T} & \text{if } 0 \leq y_n < B \\ \frac{y_n + (1 - 2 \cdot ep) \cdot T}{1 - 2 \cdot \epsilon \cdot T} & \text{if } b \leq y_n < 1 - T \\ \frac{y_n + T - 1}{1 + 2 \cdot \epsilon \cdot T} & \text{if } 1 - T \leq y_n \leq B + 1 \\ \frac{y_n + (1 - 2 \cdot \epsilon) \cdot T - 1}{1 - 2 * \epsilon \cdot T} & \text{otherwise.} \end{cases} \quad (9)$$

$$f(y_n) = \frac{1}{A} \sum_{m=1}^{4} a_m \sin(2\pi m y_n) \quad (10)$$

With $T = 0.5$, $\epsilon = 0.25$ and $B = 0.5 + (\epsilon - 1) \cdot T$ in (9) and with $a_m = \{1, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}\}$ and $A = a1 + a2 + a3 + a4$ in (10).

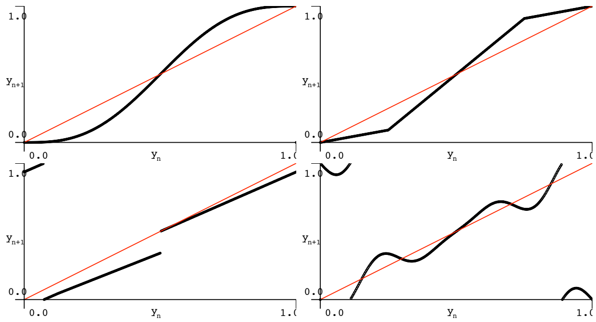

Figure 8: *Four non-linearities functions explored: Top, left: Sine, Top; right: Triangle; Bottom, left: Piecewise linear cardiorespiratory coupling model [19]; Bottom, right: truncated Fourier-series. All depicted at* $k = 1, \Omega = 0$*.*

Equation (8) is a triangle function. Equation (9) is a piecewise linear function from the biomedical literature [19] and equation (10) is a Fourier-series composition with four terms. The functions are shown in Figure 8.

For small k the main effect of variation in the two-fold. One is the precise occurrence of intersection with the fixed-point line. The second is the wave-shaping character of the function.

For large k all four example exhibit chaotic properties, though for the same parameter values, the behavior can naturally be very different. Figure 9 shows the same parameter point $\Omega = 0.2, k = 16$ for all four functions. Note that the standard circle map exhibits a stable periodic orbit, which has however a large orbit period. The triangle map shows a chaotic pattern. The cardiorespiratory model shows a stable orbit, including a dense attractor with some jitter, and the Fourier series also displays a chaotic regime.

6. SONIC RESPONSES

This section gives a subjective description of the sounds heard for most figures presented so far. To make the results accessible to the audible range, an iteration frequency of 22050 was used and the result of function (5) was downsampled by a factor of ten. The purpose of this section is to give a rough idea of the sonic character of the responses that certainly are not well captured in visual depiction. Some figures while looking very similar have rather different sounding responses.

- The example of bifurcation depicted in Figure 5 corresponds to a sine wave that successively gains new partials, eventually a high frequency noise develops, the noise becomes more broadband and eventually dominates the spectrum.
- The jitter pattern from Figure 2 is perceived as a sine wave with some underlaying noise.
- The bistable sounds of Figure 3 are two sine waves which are at a perfectly tuned fourth interval.
- The bottom two examples of the singular orbit within the noise regime sounds like a pure oscillator with some partials in the singular case, and sounds like noise with irregular rhythmic subpatterns in the chaotic case.
- The different initial conditions of the case depicted in Figure 4 is a sine wave with a low frequency pitch bending modulation for the first regular case, while the other non-chaotic orbit has a rich spectrum. The chaotic case sound like a low frequency band with a rhythmic subpattern.

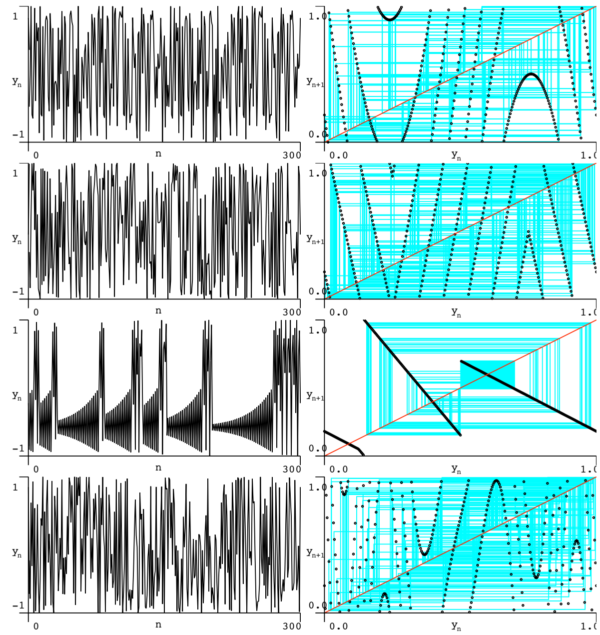

Figure 9: *The result of the general circle map for (top to bottom) sine, triangle, piecewise linear and Fourier series functions at* $\Omega = 0.2, k = 16$.

- The example of variation of the nonlinear functions of Figure 9 sound like a simple sound with a light pitch bend for the sine function, rhythmic noise for the triangle function, a single sound with complex spectrum for the piecewise linear function and broadband noise with a mild periodic pitch bend for the Fourier series.
- Other interesting examples (without visual depiction) can be found at $\Omega = 0.4, k = 1.1$ for high pitched narrow-band noise. Narrowband low-frequency noise with and without rhythmic subpatterns can be found at $\Omega = 0.36, k = 16.0$ and $k = 15.8$.

Overall it is important to note, that pure noise responses are rather rare, even "noisy"-sounding responses often have additional features, for example underlaying rhythmic patterns, they may be band-limited and be subject to pitch-bending phenomena. The noisy aspect of the sound my be mixed with pitched or narrowband sounds. Hence chaotic patterns and perceptual noise are typically not the same.

7. CONCLUSIONS

We have demonstrated a number of properties of the circle map for sound synthesis by experimentation with both the parameters of the map and by varying the non-linearity. Some of the sonic results of non-linear iterative maps have already been discussed in the context of granular synthesis [9, 10]. The advantage of considering the circle map is the transition of linear and familiar behavior into rather complex regions which exhibits varied responses, like chaotic, noisy-sounding responses, to pure and mixed sine wave, amplitude modulation, and pitch bending. One intriguing feature of these maps is their evident computational efficiency, requiring one function lookup, one multiplication, 2 additions and one additional memory lookup per time step. By the rich diversity of possible responses, and guaranteed stability this makes iterated maps attractive. The main disadvantage of the method is the difficulty of control. The purpose of this work is to prepare for eventual recommendations control of perceptually relevant responses through these parameters, which is planned.

The circle map is also attractive because by its one parameter perturbation between linear and non-linear behavior allows for classification of the method with respect to the two end of this behavioral spectrum.

8. ACKNOWLEDGMENTS

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